# DHANAMANJURI UNIVERSITY
## DECEMBER-2025

**Name of Programme** : **M.A./M.Sc. Mathematics**
**Semester** : **1st**
**Paper Code** : **MAT-505**
**Paper Title** : **Differential Equations-I**
**Full Marks** : **80**
**Pass Marks** : **32** **Duration: 3 hours**

***The figures in the margin indicate full marks for the questions.***

***Answers all the questions:***

**1. Answer any three from the following questions:** $10 \times 3 = 30$

a) Define a regular singular point. Solve the Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ in series, taking $2p$ as non-integral about the point $x = 0$. 1+9=10

b) Define an indicial equation. Solve in series the equation $x(1-x)y'' - 3xy' - y = 0$ near the point $x = 0$. 1+9=10

c) Define an equivalent integral equation of an initial value problem. Apply Picard's successive approximation method to the initial value problem $\frac{dy}{dx} = y,\ y(0) = 1$ and show that the successive approximations tend to the limit $e^x$, the exact solution. 1+9=10

d) Define Lipschitz continuity. Prove that if $f$ is continuous and satisfy a Lipschitz condition in $R = \{(x,y) : |x - x_0| \le a, |y - y_0| \le b, a > 0, b > 0\}$ and if a solution of the initial value problem: $y' = f(x,y),\ y(x_0) = y_0$ exists, then it is unique. 1+9=10

e) Transform the equation $y''(x) = f(x, y, y', \ldots, y^{(n-1)})$ into a system of first order equations. Solve the equation $y'' + 4y' + 40y = 0$, $y(0) = 1,\ y'(0) = 0$ by transforming into a system of equations. 2+8=10

**2. Answer any two from the following questions:** $10 \times 2 = 20$

a) i) Represent the following linear equation
$y^{(4)} - 6y^{(3)} + 25y^{(2)} + 7y^{(1)} + 8y = 12$ in vector matrix form.

ii) Prove that if the Wronskian of two solutions of the differential equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ where $a_0(x)$, $a_1(x)$, $a_2(x)$ are continuous and $a_0 \neq 0 \ \forall x \in (a,b)$ is linearly independent, then $W(x) = Ae^{-\int \frac{a_1(x)}{a_0(x)}dx}$. 5+5=10

b) i) Show that the Wronskian of the functions $x^2$ and $x^2 \log x$ is non-zero. Can these functions be independent solutions of an ODE? If so, determine the D.E.

ii) Prove that two solutions of the linear homogeneous second order differential equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ where $a_0(x)$, $a_1(x)$, $a_2(x)$ are continuous and $a_0 \neq 0 \ \forall x \in (a,b)$ are linearly dependent if and only if their Wronskian is identically zero. 5+5=10

c) Show that linearly independent solutions of $y'' - 2y' + 2y = 0$ are $e^x \sin x$ and $e^x \cos x$. What is the general solution? Find the solution $y(x)$ with initial condition $y(0) = 2$ and $y'(0) = 3$. 10

d) Consider the system $Y' = AY + B$ where $A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} e^x \\ e^{-x} \end{pmatrix}$.

Show that $\varphi(x) = \begin{pmatrix} e^{3x} & 2xe^{3x} \\ 0 & e^{3x} \end{pmatrix}$ is a fundamental matrix. Compute a solution $Y$ of the system with $Y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. 10

**3. Answer any three from the following questions:** $10 \times 3 = 30$

a) i) Prove that the necessary and sufficient condition for a second order linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \text{ where } a_0(x) \neq 0 \text{ on } [a,b]$$

to be self-adjoint is that $a_0'(x) = a_1(x)$ on $[a,b]$.

ii) Prove that the adjoint of $L_2^* = L_2$

iii) Find the adjoint equation of the equation

$$(2x+1)\frac{d^2y}{dx^2} + x^3\frac{dy}{dx} + y = 0 \qquad 6+2+2=10$$

b) i) Transform the second order linear differential equation $y'' + P(x)y' + Q(x)y = 0$ into the normal form $u'' + q(x)u = 0$.

ii) Find the normal form of Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ and use it to show that every non-trivial solution has infinitely many positive zeros. 5+5=10

c) Find the eigenvalues and eigenfunctions of the Sturm-Liouville system

$$y'' + \lambda y = 0, \ 0 \le x \le \pi$$
$$y(0) = 0, \ y'(\pi) = 0 \qquad 10$$

d) Find the eigenvalues and eigenfunctions of the periodic Sturm-Liouville system

$$y'' + \lambda y = 0, \ -\pi \le x \le \pi$$
$$y(-\pi) = y(\pi), \ y'(-\pi) = y'(\pi) \qquad 10$$

e) What is a periodic Sturm-Liouville system. Define eigenvalues and eigenfunctions of a Sturm-Liouville system. Prove that the eigenfunctions of a periodic Sturm-Liouville system in $[a,b]$ are orthogonal with respect to the weight function $s(x)$ in $[a,b]$. 1+2+7=10

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